Computations of Mode-dependent Dwell Times for Discrete-Time Switching System

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Abstract

Given a system that switches among $N$ linear subsystems, this paper shows an approach that computes $N$ dwell times, one for each of the subsystems, such that the overall system is stable under switching signals that respect the dwell times. The dwell times are obtained progressively starting from the groups of all pairwise switching systems, and increasing the size of the group by one for each step. In each progressive step, a bisection search algorithm is used to obtain the mode-dependent dwell times for that step. When the final step is reached, the $N$-mode dwell times are obtained. These dwell times are smaller, in terms of their sum, than an existing approach in recent literature for all the examples considered in this paper.

Key words: Switching systems, Mode-dependent dwell time, Constrained Systems, Stability

1 Introduction

This paper considers the following constrained discrete-time switching linear system:

\begin{align}
  x(t+1) &= A_{\sigma(t)} x(t), \quad (1a) \\
  x(t) &\in \mathcal{X}, \quad \forall t \in \mathbb{Z}^+ \quad (1b)
\end{align}

where $x(t) \in \mathbb{R}^n$ is the state variable and $\sigma(t) : \mathbb{Z}^+ \to \mathbb{I}_N := \{1, \ldots, N\}$ is a time-dependent switching signal that indicates the current active mode of the system among $N$ possible modes in $\mathcal{A} := \{A_1, \ldots, A_N\}$. The constraint set $\mathcal{X} \subset \mathbb{R}^n$ models physical state constraints imposed on the system, including those arising from the actuator via some appropriate feedback controller.

The study of system (1a) is quite active in the past decade, especially on the conditions that ensure stability of the system when $\sigma(\cdot)$ is an arbitrary switching function. These studies are usually based on the existence of some appropriate Lyapunov functions (Liberzon and Morse, 1999). However, most of these approaches do not consider the existence of constraints. Another research area is the determination of the dwell time needed to ensure stability. When all $A_i$’s are stable, it is well known that the stability of the origin can be ensured if the time duration spent in each subsystem is sufficiently long. Correspondingly, several upper bounds of the stabilizing dwell time needed have appeared (Chesi, Colaneri, Geromel, Middleton and Shorten, 2012; Blanchini, Casagrande and Miani, 2010; Colaneri, 2009; Zhang and Boukas, 2009) including those that provide both necessary and sufficient conditions for the characterization of the minimal dwell time (Dehghan and Ong, 2012b; Blanchini and Colaneri, 2010). Several relaxations to the dwell-time approach have also appeared. One is the use of average dwell time (Hespanha and Morse, 1999) instead of strict dwell time requirement at each switching instant. However, average-dwell-time requirement may result in the state moving far away from the origin and violate physical constraints. Another relaxation is to consider a dwell time for each mode of the system instead of one common dwell time for all (Zhao, Zhang, Shi and Liu, 2012; Blanchini et al., 2010).

Like the last relaxation, this work is concerned with the determination of the mode-dependent dwell times. It uses as a basic step the algorithm mentioned in the work of Dehghan and Ong (2012b). In that work, an algorithm is provided that computes the minimal common dwell-time for system (1). That approach, together with several properties introduced hereafter, are used to compute the mode-dependent dwell times for (1).
The rest of this paper is organized as follows. This section ends with a description of the notations used. Section 2 reviews some standard terminology and results for switching systems. Section 3.1 shows the computations of the minimal mode-dependent dwell times for a system with two modes. Section 3.2 shows the constructive procedure of computing the mode-dependent dwell times for system with increasing number of modes. Sections 4 and 5 contain, respectively, numerical examples and conclusions.

The following standard notations are used. $\mathbb{R}$ and $\mathbb{Z}^+$ are the sets of real numbers and non-negative integers respectively. The spectral radius of $A$ is denoted by $\rho(A)$. The floor function, $\lfloor a \rfloor$, is the largest integer that is less than or equal to $a$. Positive definite (semi-definite) matrix, $P \in \mathbb{R}^{n \times n}$, is indicated by $P > 0$ ($\geq 0$). When the context is clear, the set of scalar values $\{\tau_1, \cdots, \tau_N\}$ is considered as a vector, $(\tau_1, \cdots, \tau_N) \in \mathbb{R}^N$. The $p$-norm of a vector or a matrix is $\| \cdot \|_p, p = 1, 2, \infty$ with $\| \cdot \|$ refers to the 2-norm. Suppose $\alpha > 0$, $\mathcal{X} \subset \mathbb{R}^n$ is a compact set that contains 0 in its interior, then $\alpha \mathcal{X} := \{\alpha x : x \in \mathcal{X}\}$. Boldface 1 indicates the vector of all 1s and 0 indicates the empty set. $|I|$ is the cardinality of a discrete set $I$. Other notations are introduced when needed.

2 Preliminaries

This section begins with the assumptions on the system and a review of standard definitions of systems under dwell-time switching. The system (1) is assumed to satisfy the following assumptions: (A1) The spectral radius of each individual subsystem $A_i, i \in I_N$ is less than 1; (A2) The constraint set $\mathcal{X}$ is a polytope represented by $\mathcal{X} = \{x : Rx \leq 1\}$ for some appropriate matrix $R \in \mathbb{R}^{r \times n};$ (A3) $(A_i, R)$ is observable for at least one $A_i \in \mathcal{A}$. Assumption (A1) defines the family of systems considered herewith. The polyhedral assumption of (A2) is made to facilitate numerical computations. Assumption (A3) ensures the compactness of the sets. It applies to only one $i \in I_N$ since the contractive set is applicable to all admissible sequences including one where $\sigma(k) = i$ for all $k \in \mathbb{Z}^+$. Of course, if (A3) is not satisfied, system (1) can be reformulated to consider only the observable subsystem of $A_i$.

Denote the dwell times for system (1a) by

$$\Gamma := \{\tau_1, \cdots, \tau_N\}$$

where $\tau_i$ is associated with the $i^{th}$ subsystem. A few standard definitions are given next.

**Definition 1** Let any switching sequence of (1) be denoted by $S_\Gamma(t) = \{\sigma(t-1), \cdots, \sigma(1), \sigma(0)\}$ with switching instants at $t_0, t_1, \cdots, t_k, \cdots$ with $t_0 = 0$ and $t_k < t_{k+1}$. This switching sequence is said to be mode-dependent-dwell-time admissible (MD-DT admissible) if $t_{k+1} - t_k \geq \tau_i$ if $\sigma(t_k) = \sigma(t_{k+1} - 1) = i$ for all $k \in \mathbb{Z}^+$.

For notational convenience, $A_{S_\Gamma(t)}$ refers to the product $\Pi_{i=0}^{t-1}A_{\sigma(i)}$, associated with sequence $S_\Gamma(t)$.

**Definition 2** A set $\Omega \subset \mathbb{R}^n$ is said to be Dwell-Time contractive (DT-contractive) w.r.t. system (1a) and dwell-time $\Gamma$ and a contractive factor $\lambda \in (0, 1)$ if $x \in \Omega$ implies $A_{S_\Gamma(t)}x \in \lambda \Omega$ for all MD-DT admissible switching sequences $S_\Gamma(t)$.

**Definition 3** A set $\Omega \subset \mathbb{R}^n$ is said to be Constraint Admissible DT-contractive (CADT-contractive) w.r.t. system (1) with dwell-time $\Gamma$ and factor $\lambda \in (0, 1)$ if it is DT-contractive and $x(t) \in \mathcal{X}$ for all $t \in \mathbb{Z}^+$.

3 Main Results

Definition 2 requires the existence of $\Omega$ such that $A_{S_\Gamma(t)}x \in \lambda \Omega$ for all MD-DT admissible sequences $S_\Gamma(t)$. Clearly, it is not possible to find all MD-DT admissible sequences. The next theorem with proof given in the appendix shows a necessary and sufficient condition for a DT-contractive set for system (1). To state it formally, let

$$\mathcal{T}_i := \{\tau_i, \tau_i + 1, \ldots, 2\tau_i - 1\} \forall i \in I_N; \quad \mathcal{T}_G := \cup_{i \in I_N} \mathcal{T}_i.$$  \hspace{1cm} (3)

**Theorem 1** Suppose (A1) is satisfied and $\mathcal{T}_i$ and $\mathcal{T}_G$ given by (2) and (3) respectively. A non-empty set $\Omega \subset \mathbb{R}^n$ is DT-contractive with contraction factor $\lambda \in (0, 1)$ for system (1a) with dwell times $\Gamma$, if and only if for every $x \in \Omega$,

$$A_i^\ell x \in \lambda \Omega \quad \text{for all } t \in \mathcal{T}_i \text{ and for all } i \in I_N.$$  \hspace{1cm} (4)

An example that illustrates Theorem 1 is now given. Consider $\mathcal{A} = \{A_1, A_2\}, \quad \Gamma = (\tau_1, \tau_2) = (3, 2), \quad \mathcal{T}_i = (3, 4, 5)$ and $\mathcal{T}_G = (2, 3)$. Suppose an admissible sequence is such that $x(23) = A_{S_\Gamma(23)}x_0 = A_1^8 A_2^4 A_1^3 x_0$. Using the result of Theorem 1 and by decomposing $S_\Gamma(t)$ as a concatenation of appropriate MD-DT admissible subsequences, $x(23) = [A_1^3 A_2^1] [A_2^2 A_2^1] [A_1^1]^2 A_1^3] x_0 \in \lambda^3 \Omega$, since $A_i^\ell x_0 \in \lambda \Omega$ for all $t \in \{3, 4, 5\}$ and $A_2^4 x_0 \in \lambda \Omega$ for all $t \in \{2, 3\}$ if $x_0 \in \Omega$.

The computation of such an $\Omega$ set is described next. For this purpose, let $P^i(\Omega) := \{x : A_i x \in \Omega\}$ be the set of $x$ that can be brought into $\Omega$ by system $A_i$ in one time step. Repeating this process $\ell$ times lead to $P^\ell(\Omega) := P^i(\Omega) = \{x : A_i^\ell x \in \Omega\}$ and is referred to as the $\ell$-step backward set of $\Omega$ under system $A_i$. Define

$$P^i(\Omega) := \cap_{t \in \mathcal{T}_i} P^t_i(\Omega)$$  \hspace{1cm} (5)
as the intersection of \( \hat{P}^i(\Omega) \) for \( \ell = \tau_1, \cdots, 2\tau_i - 1 \) for \( A_i \). With this definition, the algorithm for computing the maximal DT-contractive set for a particular choice of \( \lambda \in (0, 1) \), denoted by \( O_\infty^\lambda \), is now given.

**Algorithm 1** Computation of polyhedral DT-contractive set

**Input:** \( I_N, \Gamma, A \) and \( \lambda \).

**Output:** \( O_\infty^\lambda \).

(i) Set \( k = 0 \) and let \( O_0^k := A \).

(ii) Compute \( P^i(\lambda O_k^\lambda) \) for every \( i \in I_N \) and let \( O_{k+1}^\lambda := O_k^\lambda \bigcap_{i \in I_N} P^i(\lambda O_k^\lambda) \).

(iii) If \( O_{k+1}^\lambda \equiv O_k^\lambda \) set \( O_{\infty}^\lambda = O_k^\lambda \) then stop, else set \( k = k + 1 \) and goto step (ii).

Step (ii) imposes the condition of Theorem 1 with \( P^i(\lambda) = \bigcap_{\ell \in T} \hat{P}^i(\lambda) \) as given by (5). For mode \( i \), the \( P \) operator is applied for \( \tau_i, \cdots, 2\tau_i - 1 \) to obtain the set \( P^i(\lambda) \) such that points starting from it will return to \( \lambda P^i(\lambda) \) after \( \tau_i \) to \( 2\tau_i - 1 \) steps. Obviously, the \( O_\infty^\lambda \) obtained using the above algorithm depends on the choices of \( A, X, \lambda \) and \( \tau \). For notational convenience, such dependencies are not shown unless warranted.

An important pertinent question is the stability of the switching system under admissible MD-DT switching sequences if Algorithm 1 obtains an empty \( \Omega \) set for all \( \lambda < 1 \). This is answered in the form of a theorem given below with proof given in the appendix. Note that the stability as defined in the statement of the theorem is equivalent to uniform stability with respect to all MD-DT admissible switching sequences.

**Theorem 2** Suppose (A1)-(A3) are satisfied with \( \Gamma \) as defined above. The origin of system (1) is asymptotically stable under \( \Gamma \) if and only if Algorithm 1 yields a non-empty DT-contractive set with a contraction factor \( \lambda \in (0, 1) \).

The remainder of this section describes a procedure that computes \( \Gamma \) with an associated DT-contractive set \( \Omega \) in the sense of Theorem 1. The intention is to find \( \Gamma \) having the smallest \( \| \Gamma \|_1 \) of \( (\sum_{i=1}^{t_i} \tau_i) \). Since \( \{\tau_c, \cdots, \tau_c\} \) is an admissible \( \Gamma \), the search of \( \Gamma \) is therefore restricted to

\[ Y_c := \{ (\tau_1, \cdots, \tau_N) : \tau_i \leq \tau_c \text{ for all } i \in I_N \}. \tag{6} \]

As there can be more than one choice of \( \Gamma \) having the same value of \( \| \Gamma \|_1 \), the following definition of optimal dwell time is meant to accept any one of them.

**Definition 4** Given system (1a) and an optimal common dwell time \( \tau_c \), such that \( Y_c \) is defined by (6). The system has an optimal mode-dependent dwell time \( \tau_c \) if \( \Gamma \) has the smallest value of \( \sum_{i=1}^{N} \tau_i \) and the origin of system (1a) is asymptotically stable under any MD-DT switching sequence, \( \mathcal{S}_1(t) \).

The next proposition contains some useful observations associated with the definition of \( Y_c \), that facilitates the search for the minimal \( \Gamma \). In particular, property (ii) is used to determine the extreme values of \( \Gamma \) for the bisection method discussed in the sequel.

**Proposition 1** (i) Suppose \( \Gamma_a = \{\tau_1, \cdots, \tau_N\} \) is a stabilizing dwell time for system (1a), so is \( \Gamma = \{\tau_1, \cdots, \tau_N\} \) with \( \tau_i \geq \tau_c \) for all \( i \in I_N \). (ii) Suppose a common minimal dwell time \( \tau_c \) is known for system (1a) and \( \tau_c \) is that defined by (6). Let \( \Gamma \subseteq Y_c \) be a stabilizing dwell time such that the origin of system (1a) is stable. Then, at least one \( \tau_i \in \Gamma \) is equal to \( \tau_c \).

**PROOF.** (i) Obvious. (ii) Suppose the assertion is not true. This means that there exists a \( \hat{\Gamma} = \{\hat{\tau}_1, \cdots, \hat{\tau}_N\} \) with \( \hat{\tau}_i < \tau_c \) for all \( i \in I_N \) such that system (1a) is stable under admissible dwell-time switching sequences. Let \( \hat{\tau}_{\max} = \max_{i \in I_N} \hat{\tau}_i \). By (i), it follows that \( \{\hat{\tau}_1, \cdots, \hat{\tau}_{\max}\} \) is a stabilizing mode-dependent dwell time. Since \( \hat{\tau}_{\max} < \tau_c \), this contradicts \( \tau_c \) being the minimal common dwell time. \( \Box \)

This search of a stabilizing dwell time is facilitated by a bisection algorithm with Algorithm 1 as a subalgorithm. The bisection algorithm has inputs: an index set \( I \subseteq I_N \), \( \{A_i : i \in I\}, X, \lambda = 1 - \epsilon \) for some small \( \epsilon > 0 \), \( \Gamma := \{\tau_i : i \in I\} \) and \( \Gamma := \{\tau_i : i \in I\} \). Here, \( \Gamma \) is a non-stabilizing dwell time while \( \Gamma \) is a stabilizing dwell time.

**Algorithm 2** Bisection Algorithm

**Input:** \( I, \{A_i : i \in I\}, X, \lambda \) and \( \Gamma \).

**Output:** A stabilizing dwell time \( \Gamma \).

(i) Let \( \Gamma = \{\tau_i : i \in I\} \) where \( \tau_i = \frac{\tau_c + \tau_c}{2} \). If \( \Gamma = \Gamma \), set \( \Gamma = \Gamma \) and terminate.

(ii) Invoke Algorithm 1 with inputs \( I, \Gamma, \lambda \) and \( X \).

(iii) If \( O_\infty^\lambda = \emptyset \), set \( \Gamma = \Gamma \). Otherwise, set \( \Gamma = \Gamma \). Goto step (i).

The choice of \( I \) and the initial values of \( \Gamma \) and \( \Gamma \) are discussed next. The choice of \( I \) is chosen progressively from all pairwise groupings of \( I_N \) and incrementing them one at a time. The choice of \( \Gamma \) and \( \Gamma \) are chosen based on \( \tau_c \) of each \( I \). The basic idea is first described in the exemplary case of a two-mode system.

### 3.1 System with two modes

Given system (1a) with \( I_N = \{1, 2\} \) and the common minimal dwell time, \( \tau_c \), for this system. Let \( \Gamma = \{\tau_c, 0\} \) and \( \Gamma = \{\tau_c, \tau_c\} \). Invoke the bisection algorithm using \( \Gamma \), \( I = I_N \), \( A = \{A_1, A_2\} \) and \( X \). Let the solution be \( \Gamma_1 \). Repeat the above but with \( \Gamma = \{0, \tau_c\} \) and \( \Gamma = \{\tau_c, \tau_c\} \) and let the solution be \( \Gamma_2 \). The optimal dwell time is
the one having the smaller total dwell times, given by \( \min\{\tau_1 + \tau_2; (\tau_1, \tau_2) \in \Gamma_i, i = 1, 2\} \). The setting of \( \Gamma = \{\tau_0, 0\} \) (or \( \Gamma = \{0, \tau_1\} \)) in the above is a notation to denote that \( \Gamma \) is an inadmissible dwell time for the proper working of the bisection Algorithm (note that step (i) of Algorithm 2).

**Lemma 1.** The mode-dependent dwell time obtained using the above procedure is optimal (in the sense of Definition 4) for system (1a) with \( \mathcal{I}_N = \{1, 2\} \).

**Proof.** The result is based on the fact that Algorithm 1 is both necessary and sufficient for \( \Gamma \) being a stabilizing dwell time, property (ii) of Proposition 1 and that \( \tau_1 + \tau_2 \) is the smaller of \( \Gamma_1 \) and \( \Gamma_2 \). \( \square \)

As an example, consider the 2-mode system with \( A_1 = [1, 0.1; -0.2, 0.9], A_2 = [1, 0.1; -0.9, 0.8] \) and \( \mathcal{X} = [x \in \mathbb{R}^2 : \|x\|_\infty \leq 1] \). The values of \( \Gamma \) are given in the following table with the optimal \( \Gamma = \{1, 7\} \).

<table>
<thead>
<tr>
<th>( \mathcal{I} )</th>
<th>( \Gamma )</th>
<th>( \Gamma_{\text{out}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {1, 2} )</td>
<td>( 1 )</td>
<td>( 7 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 7 )</td>
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<tr>
<td>( 2 )</td>
<td>( 7 )</td>
<td>( 1 )</td>
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<td>( 7 )</td>
<td>( 7 )</td>
<td>( 7 )</td>
</tr>
</tbody>
</table>

Table 1
Illustrative example with 2 modes

### 3.2 System with more than two modes

The main idea for this case is first described using a simple example of a 3-mode system, followed by a more formal description for the general case. Let \( \mathcal{I}_N = \{1, 2, 3\} \). The basic idea is to consider the common dwell time \( \tau_c \) and all optimal dwell times from the 2-mode pairwise results to identify the \( \Gamma_1 \) and \( \Gamma_2 \). Consider the example of \( A_3 = [0.95, 0.09; -0.94, 0.86] \) with \( A_1, A_2 \) and \( \mathcal{X} \) being that from the previous example. The \( \tau_c \) (obtained using approach from Dehghan and Ong (2012b)) for this 3-mode system is 8. There are altogether 3 possible combinations of \( \tau_c \); each one corresponding to \( \tau_i \) for \( i = 1, 2, 3 \). In addition, the optimal 2-mode dwell times for systems \( \{1, 2\} \), \( \{1, 3\} \) and \( \{2, 3\} \) are obtained (using the procedure described in subsection 3.1) as \( (1, 7) \), \( (1, 8) \) and \( (1, 1) \). Table 2, shows the values of \( \Gamma \) and \( \Gamma_{\text{out}} \) used with the optimal shown in bold.

<table>
<thead>
<tr>
<th>( \text{Index } j )</th>
<th>( \mathcal{I} = {1, 2, 3} )</th>
<th>( \Gamma )</th>
<th>( \Gamma_{\text{out}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 8 )</td>
<td>( 1 )</td>
<td>( 8 )</td>
</tr>
<tr>
<td>2</td>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 8 )</td>
</tr>
<tr>
<td>3</td>
<td>( \text{(not)} )</td>
<td>( 1 )</td>
<td>( 7 )</td>
</tr>
</tbody>
</table>

Table 2
Illustrative example with 3 modes

† The symbol “-” in Tables 2 and 4 indicates that the bisection Algorithm is not invoked since \( \tau_c \) is the only \( \tau \) in

The overall scheme is now described. Following the preceding examples, two procedures are needed prior to the computation of the dwell times. The first is the common dwell time, \( \tau_c \), for the \( N \)-mode system and the second is the dwell times of all \( K \)-mode combinations for \( K = 2, 3, \cdots, N - 1 \). These dwell times for the \( K \)-mode system are generated incrementally starting from \( K = 2 \), one step at a time. For every value of \( K \), there are \( C_N^2 \) distinct choices to pick \( K \) modes from the \( N \)-mode system. To compute the dwell times for the \( N \)-mode system, the stabilizing dwell times for all the \( (N-1) \)-mode systems are needed and there are \( C_N^{N-1} \) of such systems. (The dwell times of the \( (N-1) \)-mode system are, in turn, obtained from the \( (N-2) \)-mode systems.) Let the indices of these \( (N-1) \)-mode systems be identified by \( \mathcal{J}_1, \mathcal{J}_2, \cdots, \mathcal{J}_r \) where \( r = C_N^{N-1} \) with \( |\mathcal{J}_r| = N - 1 \) for all \( \ell = 1, \cdots, r \) and their dwell times be \( \Gamma_{\mathcal{J}_1}, \cdots, \Gamma_{\mathcal{J}_r} \). The key step to the scheme is in choosing the correct \( \Gamma \) and \( \Gamma \) and the procedure of doing so is given in the algorithm below.

**Algorithm 3** Computation of \( \Gamma_{\mathcal{J}} \)

**Input:** \( \{\Gamma_{\mathcal{J}}, \mathcal{J} = \mathcal{J}_1, \cdots, \mathcal{J}_r\} \), \( \tau_c \), \( \Gamma = \{\tau_{c_1}, \cdots, \tau_{c_N}\} \) and \( \mathcal{I}_N = \{i_1, \cdots, i_N\} \).

**Output:** A stabilizing dwell time \( \Gamma_{\mathcal{J}_N} \) for the corresponding \( N \)-mode system.

(i) Set \( j = 1 \).

(ii) Let \( \mathcal{I}_j = \mathcal{I}_N \setminus \mathcal{J}_j \) and let \( \mathcal{J}^* \) denote the \( \mathcal{J} \) index set (among \( \mathcal{J}_1, \cdots, \mathcal{J}_r \)) that has the same elements as \( \mathcal{J}_j \).

(iii) Let \( \Gamma_{\mathcal{J}^*} = \{\tau_{i_1}, \cdots, \tau_{i_N}\} \) where \( \tau_{i_r} = \tau_c \) and the rest of the \( \tau_i \) are set equal to those from \( \Gamma_{\mathcal{J}^*} \).

(iv) Invoke Algorithm 1 with \( \mathcal{I}_N \) and \( \Gamma_{\mathcal{J}_N} \) (in place of \( \Gamma \)) whose output is \( \mathcal{O}_N \).

(v) If \( \mathcal{O}_N = \emptyset \), set \( \mathcal{G} = \Gamma_{\mathcal{J}} \) and \( \Gamma_{\mathcal{J}_N} = \Gamma_{\mathcal{J}_N} \). Else, let \( \Gamma_{\mathcal{J}_N} = \Gamma_{\mathcal{J}_N} \cdot \mathcal{O}_N \) except that \( \tau_{i_j} \) = 0. Set \( \mathcal{G} = \Gamma_{\mathcal{J}_N} \) and \( \Gamma_{\mathcal{J}_N} = \Gamma_{\mathcal{J}_N} \).

(v) Invoke the bisection algorithm with \( \mathcal{G} \) and \( \Gamma \). Let \( \Gamma_{\mathcal{J}_N} \) be the dwell time obtained from the bisection algorithm. If \( j < N \), set \( j = j + 1 \) and goto step (i).

(v) Let \( k = \arg \min_{\mathcal{J}_N} \min\{\tau_{i_1}, \cdots, \tau_{i_N} : \Gamma_{\mathcal{J}_N} \cdot j, \cdots, \mathcal{N} \} \) and set \( \Gamma_{\mathcal{J}_N} = \Gamma_{\mathcal{J}_N} \).

Stop.

The choice of \( \Gamma_{\mathcal{J}_N} \) in step (ii) is motivated from property (ii) of Proposition 1. The rest of \( \tau_i \) are set to those from \( \Gamma_{\mathcal{J}_N} \) because they are stabilizing dwell time for the \( \mathcal{J}^* \) system.

**Lemma 2** Suppose that \( \Gamma_{\mathcal{J}} \) are the optimal dwell times for all the \((N-1)\)-mode systems whose index sets are \( \mathcal{J}_1, \cdots, \mathcal{J}_r \). Suppose the output set \( \mathcal{O}_N \) of step (ii) for the optimal index \( k \) is non-empty (k as defined by step (v)), then the corresponding \( \Gamma_{\mathcal{J}_N} \) is the optimal dwell time in the sense of Definition 4.

\( \Gamma_{\mathcal{J}_N} \) having a value of \( \tau_c \). In this case, the bisection algorithm can be avoided by setting \( \Gamma_{\mathcal{J}_N} = \Gamma_{\mathcal{J}_N} \) following the result (ii) of Proposition 1.
PROOF. Note that when \( O_{\infty} \) of step (ii) is non-empty, \( \Gamma^+_{\mathcal{I}} \) is obtained from the bisection algorithm with \( \Gamma^+_{\mathcal{I}} = \{ \tau_1, \ldots, \tau_c, \ldots, \tau_t, \ldots, \tau_{\mathcal{I}_N} \} \) and \( \Gamma^-_{\mathcal{I}} = \{ \tau_1, \ldots, 0, \ldots, \tau_{\mathcal{I}_N} \} \). Hence, \( \Gamma^+_{\mathcal{I}} \) takes the form \( \{ \tau_1, \ldots, \tau_j, \ldots, \tau_{\mathcal{I}_N} \} \). Now suppose the assertion is not true and there exists a \( \tau^* \) such that \( \tau_\ell < \tau^* \) for some \( \ell \), satisfying \( 1 \leq \ell \leq N \) such that \( \Gamma := \{ \tau_1, \ldots, \tau_\ell, \ldots, \tau_{\mathcal{I}_N} \} \) is a stabilizing dwell time. If \( \ell = j \), this leads to a contradiction since \( \tau_j \) is already the optimal under the bisection algorithm. If \( \ell \neq j \), this also leads to a contradiction since \( \Gamma \setminus \{ \tau_j \} \) is the optimal dwell time for the \((N - 1)\)-mode system. Finally, the proof is complete since the choice of \( j \) has the smallest value of \( \sum_{i=1}^{N} \tau_i \).

Remark 1 For small values of \( N \), Lemma 2 is not as conservative as it may appear as the dwell times obtained for the \( K = 2 \) are optimal following Lemma 1. The conservatism increases with higher values of \( N \). In all our numerical examples involving examples up to \( N = 5 \), the index \( j \) that achieves the minimal given in step (v) of algorithm 3 always happens under the settings (non-emptiness of \( O_{\infty} \) at step (ii)) described in Lemma 2.

4 Numerical Examples

The algorithms described in the prior sections are illustrated using two numerical examples and the results are also compared with the results of a recent publication (Zhao et al., 2012). The first example considered has \( A_4 = [0.99, -0.04; 0.4, 0.95] \) while \( A_1, A_2, A_3 \) and \( X \) are those given in the previous section. Note that the corresponding spectral radii are 0.959, 0.943, 0.950, 0.978 respectively.

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( {1, 2} )</th>
<th>( {1, 3} )</th>
<th>( {1, 4} )</th>
<th>( {2, 3} )</th>
<th>( {2, 4} )</th>
<th>( {3, 4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau_0 )</td>
<td>( \tau_c )</td>
<td>( \tau_5 )</td>
<td>( \tau_{10} )</td>
<td>( \tau_{15} )</td>
<td>( \tau_{20} )</td>
<td>( \tau_{25} )</td>
</tr>
</tbody>
</table>

Table 3: Intermediate mode-dependent dwell times for all 2-mode subsystems of Example I.

Table 3 shows the intermediate mode-dependent dwell times computed for all 2-mode subsystems as described in section 3.1. The minimal dwell times are indicated in bold font. They are used to compute the mode-dependent dwell time for the 3-mode subsystems shown in Table 4 according to the procedure described in Algorithm 3.

The result of \( K = 3 \) is shown in Table 4. Each of the 3 rows corresponds to a particular choice of \( j \) for \( j = 1, 2, 3 \) described in step (i) of Algorithm 3. Also, the \( \Sigma, \Gamma \) are those set according to step (iii) of the same Algorithm. The results of Table 4 are used to compute the results of Table 5 which shows the best mode-dependent dwell times being \( \Gamma = (15, 6, 1, 1) \) and it is optimal because the conditions of Lemma 2 is satisfied.

As a comparison, the procedure for mode-dependent dwell time is computed using the recent approach of Zhao et al. (2012). They show that if there exist \( P_i \geq 0, 0 < \lambda_i < 1 \), and \( \mu_i \geq 1 \) for each \( i \) in \( \mathcal{I}_N \) such that \( \lambda_i P_i A_i \geq P_i A_i - P_i \leq \lambda_i P_i \forall i \in \mathcal{I}_N \) and \( P_i \leq \mu_i P_i \forall (i, j) \in \mathcal{I}_N \times \mathcal{I}_N \), then the switching system is asymptotically stable under dwell times \( \tau_i \geq \frac{\ln(1/\lambda_i)}{\mu_j - \lambda_j} \forall i \in \mathcal{I}_N \). Clearly, \( \tau_i \) can be minimized by maximizing \( \lambda_i \) and minimizing \( \mu_i \), respectively.

5 Conclusions

This paper proposes an algorithmic approach to the determination of mode-dependent dwell times of a system switching among \( N \) linear subsystems. The approach builds up progressively by computing the mode-dependent dwell times of the \( K \)-mode subsystems for \( K = 2, \ldots, N \). The \( K \)-mode dwell times provide necessary conditions for the stabilizing dwell times for the \((K + 1)\)-mode subsystems and, under appropriate conditions, sufficient conditions for the optimal
mode-dependent dwell times. In the numerical examples considered, where some of $N$ modes having spectral radii close to 1, the approach yields the optimal mode-dependent dwell times that are significantly smaller than the results of a recent work in the literature.

A Appendix

Proof of Theorem 1:  
($\Rightarrow$): The solution of (1) under an admissible switching sequence $S(t)$ at time $t$ is $x(t) = A_{S(t)} x(0)$ where

$$A_{S(t)} = \cdots A_{i_t}^k \cdots A_{i_{\tau_j}}^k A_{i_0}^k$$  \hspace{1cm} (A.1)

and $k_j \geq \tau_j$ due to the MD-DT requirement. Consider any of the $A_{i_{\tau_j}}^k$ on the right hand side of (A.1). This term can be rewritten as $A_{i_{\tau_j}}^k = (A_{i_{\tau_j}}^0)_{q_{i_{\tau_j}}} A_{i_{\tau_j}}^{n_{\tau_j} - q_{i_{\tau_j}}}$, where $q_i = \lfloor \frac{k_j}{\tau_j} \rfloor$ and the superscript $k_j - q_{i_{\tau_j}}$ assumes a value from $T_i$. Now, consider the rightmost term of (A.1). Since for every $x \in \Omega$, $A_{i_t}^k x \in \lambda \Omega$ for all $t \in T_i$ and for all $A_i \in \mathcal{A}$, it follows that $A_{i_0}^{k_0 - q_{i_0}} x \in \lambda \Omega$ for any $x \in \Omega$. Similarly, $(A_{i_0}^{k_0})_{q_{i_0}} A_{i_0}^{n_{i_0} - q_{i_0} - 1} x \in \lambda^{n_{i_0} + 1} \Omega \subseteq \lambda \Omega$ as $A_{i_0}^n \Omega \subseteq \lambda \Omega$ from (4). Repeating this process for the rest of the terms in (A.1) completes the proof. 

($\Leftarrow$) Suppose $\Omega$ is DT-contractive with contraction $\lambda$, but there exists a $t \in T_i$ and some $A_i \in \mathcal{A}$ such that $A_i^k \Omega \not\subseteq \lambda \Omega$. The sequence $S(t) = \{i, i, \ldots, i\}$, which is admissible, violates the DT-contractivity of $\Omega$.

Proof of Theorem 2:  
($\Rightarrow$) Consider any admissible MD-DT sequence of the form (A.1). The existence of DT-contractive $\Omega$, implies that for every $x_0 \in \Omega$, $x(t) = A_{\hat{S}(t)} x_0 \in \lambda \Omega$ where $\hat{k} := \lfloor k_0 / \tau_0 \rfloor + \cdots + \lfloor k_n / \tau_n \rfloor$. Since $\lambda \in (0, 1)$ and $k \to \infty$ as $t \to \infty$, asymptotic stability of (1) follows. 

($\Leftarrow$) Asymptotic stability of (1) under MD-DT is equivalent to asymptotic stability of $\hat{x}(t + 1) = \hat{A} \hat{x}(t)$, $\hat{A} \in \{A_{i_t}^k : r \in T_i, i \in I_N\}$ which in turn is equivalent to the existence of a convex norm $\| \cdot \|_\sigma$ and a $\lambda^* \in (0, 1)$ such that $\| A_{i_t}^k \|_{\sigma} < \lambda^* < 1, \forall \in I_N, \forall r \in T_i$ (Dehghan and Ong, 2012a). Let $O_k := \{x : R_j x \leq 1 \text{ for all } j \in J\}$. When $O_k$ is incremented to $O_k$ in step (ii) of Algorithm 1 with $\lambda > \lambda^*$, additional inequalities are added in the form of $R_j A_{i_1}^k \cdots A_{i_j}^k x \leq \lambda^k$, where $t_j \in T_i$ for $j = 1, \ldots, k$. The remainder part of the proof shows that these added inequalities are redundant to $O_k$ after some sufficiently large $k$. For inequalities added at the $k$-th iteration of algorithm, there exist some positive real numbers $\delta_1$ and $\delta_2$ such that 

$$R_j (A_{i_1}^k \cdots A_{i_j}^k) x \leq \max_{\zeta \in \mathcal{B}(\| A_{i_1}^k \cdots A_{i_j}^k \|_{\sigma})} \delta_1 R_j \zeta$$

$$\leq \max_{\zeta \in \mathcal{B}(\| x \|_{\sigma})} \lambda^k \delta_1 \delta_2 R_j \zeta \leq \max_{\eta \in \mathcal{B}(\| x \|_{\sigma})} \lambda^k \delta_1 \delta_2 R_j \eta$$

$$\leq \lambda^k \delta_1 \delta_2 \| x \| \| R_j \| \leq \lambda^k$$

where the last inequality follows from $\lambda^* < \lambda \leq 1$. Therefore, all new inequalities added are redundant to $O_k$ and the iteration converges, yielding a non-empty $O_k$. That the above argument holds for all $\lambda \in (\lambda^*, 1)$ completes the proof.  

References


